۱-۷-۲: دیورژانس Divergence

Vector field $\vec{V}(x_1,x_2,x_3)$

$$div\vec{V}=\vec{\nabla}\cdot\vec{V}=\left(\frac{\partial}{\partial x_j}\hat{e}_j\right)\cdot\left(v_i\hat{e}_i\right)=\frac{\partial v_i}{\partial x_j}\left(\hat{e}_j\cdot\hat{e}_i\right)=\delta_{ji}\frac{\partial v_i}{\partial x_j}=\frac{\partial v_i}{\partial x_i}=v_{i,i}$$

$$div\vec{V}=tr\left(\vec{\nabla}\vec{V}\right)=tr(grad\vec{V})$$

دیورژانس میدان برداری، یک عددی می‌شود.

$$div\vec{V} = \vec{\nabla}\cdot\vec{V} = \frac{\partial v_i}{\partial x_i}$$

دیورژانس، مرتبه‌ی تانسور را یکی کاهش می‌دهد. دیورژانس میدان تانسوری، یک بردار می‌شود.

Tensor field $\tilde{T}(x_1, x_2, x_3)$

اگر بردار $\vec{a}$ ثابت باشد، رابطه‌ی زیر برقرار است.

$$\left(div\tilde{T}\right)\cdot\vec{a} = div\left(\tilde{T}^T\vec{a}\right)$$

$$div\tilde{T} = \vec{b} = b_i\hat{e}_i$$

$$b_i = \vec{b}\cdot\hat{e}_i = \left(div\tilde{T}\right)\cdot\hat{e}_i = div\left(\tilde{T}^T\hat{e}_i\right) = div\left(T^T_{ji}\hat{e}_j\right) = \vec{\nabla}\cdot\left(T_{ij}\hat{e}_j\right)$$

$$= \left(\frac{\partial}{\partial x_m}\hat{e}_m\right)\cdot\left(T_{ij}\hat{e}_j\right) = \delta_{mj}\frac{\partial T_{ij}}{\partial x_m} = \frac{\partial T_{ij}}{\partial x_j} \Rightarrow div\tilde{T} = \frac{\partial T_{ij}}{\partial x_j}\hat{e}_i = T_{ij,j}\hat{e}_i$$

$$div\tilde{T} = \frac{\partial T_{ij}}{\partial x_j}\hat{e}_i$$

$$div\vec{V} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

$$div\tilde{T} = \begin{pmatrix} \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} \\ \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} \\ \frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} \end{pmatrix}$$

مثال: حاصل عبارت‌های زیر را به‌دست آورید.

1) $div\left(\alpha\vec{a}\right)$

$$div\left(\alpha\vec{a}\right)=\frac{\partial\left(\alpha a_i\right)}{\partial x_i}=\alpha\frac{\partial a_i}{\partial x_i}+\frac{\partial\alpha}{\partial x_i}a_i=\delta_{ij}\alpha\frac{\partial a_j}{\partial x_i}+\delta_{ij}\frac{\partial\alpha}{\partial x_i}a_j$$

$$=\alpha\left(\frac{\partial}{\partial x_i}\hat{e}_i\right)\cdot\left(a_j\hat{e}_j\right)+\left(\frac{\partial\alpha}{\partial x_i}\hat{e}_i\right)\cdot\left(a_j\hat{e}_j\right)=\alpha(\vec{\nabla}\cdot\vec{a})+\vec{a}\cdot(\vec{\nabla}\alpha)$$

$$div\,(\alpha\vec{a})=\alpha\,div\vec{a}+\vec{a}\cdot(\vec{\nabla}\alpha)$$

2) $div\left(\alpha\tilde{T}\right)$

$$div\left(\alpha\tilde{T}\right)=\frac{\partial\left(\alpha T_{ij}\right)}{\partial x_j}\hat{e}_i=\alpha\frac{\partial T_{ij}}{\partial x_j}\hat{e}_i+\frac{\partial\alpha}{\partial x_j}T_{ij}\hat{e}_i=\alpha\,div\tilde{T}+\tilde{T}\frac{\partial\alpha}{\partial x_j}\hat{e}_j$$

$$=\alpha\,div\tilde{T}+\tilde{T}\,(grad\,\alpha)$$

$$div\,(\alpha\tilde{T})=\alpha\,div\tilde{T}+\tilde{T}\,(\vec{\nabla}\alpha)$$

۱-۷-۳: کرل Curl

Vector field $\vec{V}(x_1,x_2,x_3)$

$$curl\vec{V}=\vec{\nabla}\times\vec{V}=\left(\frac{\partial}{\partial x_j}\hat{e}_j\right)\times\left(v_i\hat{e}_i\right)=\frac{\partial v_i}{\partial x_j}\left(\hat{e}_j\times\hat{e}_i\right)=-\varepsilon_{ijk}\frac{\partial v_i}{\partial x_j}\hat{e}_k$$

$$curl\vec{V}=\vec{\nabla}\times\vec{V}=-\varepsilon_{ijk}\frac{\partial v_i}{\partial x_j}\hat{e}_k$$

تمرین: نشان دهید که کرل یک میدان برداری، بردار دوگان بخش پادمتقارن گرادیان آن میدان برداری است.

$$\vec{\nabla}\vec{V}=\left(\vec{\nabla}\vec{V}\right)^S+\left(\vec{\nabla}\vec{V}\right)^A\ ,\ \left(\vec{\nabla}\vec{V}\right)^A=\frac{1}{2}\left[\left(\vec{\nabla}\vec{V}\right)-\left(\vec{\nabla}\vec{V}\right)^T\right]$$

$$t_k^A=-\frac{1}{2}\varepsilon_{ijk}\left(\vec{\nabla}\vec{V}\right)_{ij}^A\ \rightarrow\ \vec{t}^A=-\frac{1}{2}\varepsilon_{ijk}\left(\vec{\nabla}\vec{V}\right)_{ij}^A\hat{e}_k=\frac{1}{2}curl\vec{V}\ \Rightarrow\ curl\vec{V}=2\vec{t}^A$$

کرل فقط برای میدان برداری تعریف می‌شود.

$$curl\vec{V}=\begin{pmatrix}\dfrac{\partial v_3}{\partial x_2}-\dfrac{\partial v_2}{\partial x_3}\\ \dfrac{\partial v_1}{\partial x_3}-\dfrac{\partial v_3}{\partial x_1}\\ \dfrac{\partial v_2}{\partial x_1}-\dfrac{\partial v_1}{\partial x_2}\end{pmatrix}$$

۱-۷-۴: لاپلاسین Laplacian

به دیورژانس گرادیان یک میدان، لاپلاسین می‌گویند.

$$laplacian=div\,(grad\,)\neq grad\,(div\,)$$

$$\vec{\nabla}\cdot\vec{\nabla}=\nabla^2=\left(\frac{\partial}{\partial x_i}\hat{e}_i\right)\cdot\left(\frac{\partial}{\partial x_j}\hat{e}_j\right)=\delta_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}=\frac{\partial^2}{\partial x_i\partial x_i}$$

<u>Scalar field</u> $\phi=\phi(\vec{r})$

$$\nabla^2\phi=\vec{\nabla}\cdot(\vec{\nabla}\phi)=div\,(\vec{\nabla}\phi)=\frac{\partial^2\phi}{\partial x_i\partial x_i}=\phi_{,ii}$$

<u>Vector field</u> $\vec{V}=\vec{V}(\vec{r})$

$$\nabla^2\vec{V}=div\,(\vec{\nabla}\vec{V})=div\left(\frac{\partial v_i}{\partial x_j}\hat{e}_i\circ\hat{e}_j\right)=\frac{\partial}{\partial x_k}\left(\frac{\partial v_i}{\partial x_j}\hat{e}_i\circ\hat{e}_j\right)\hat{e}_k$$

$$=\frac{\partial^2 v_i}{\partial x_k\partial x_j}\left(\hat{e}_i\circ\hat{e}_j\right)\hat{e}_k=\frac{\partial^2 v_i}{\partial x_j\partial x_k}\left(\hat{e}_j\cdot\hat{e}_k\right)\hat{e}_i=\delta_{jk}\frac{\partial^2 v_i}{\partial x_j\partial x_k}\hat{e}_i=\frac{\partial^2 v_i}{\partial x_j\partial x_j}\hat{e}_i=v_{i,jj}\hat{e}_i$$

$$\nabla^2\phi=\frac{\partial^2\phi}{\partial x_i\partial x_i}\quad\&\quad\nabla^2\vec{V}=\frac{\partial^2 v_i}{\partial x_j\partial x_j}\hat{e}_i$$

$$V^{\,2}\phi=\frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}+\frac{\partial^2\phi}{\partial x_3^2}$$

$$\nabla^2\vec{V}=\begin{pmatrix}\dfrac{\partial^2 v_1}{\partial x_1^2}+\dfrac{\partial^2 v_1}{\partial x_2^2}+\dfrac{\partial^2 v_1}{\partial x_3^2}\\ \dfrac{\partial^2 v_2}{\partial x_1^2}+\dfrac{\partial^2 v_2}{\partial x_2^2}+\dfrac{\partial^2 v_2}{\partial x_3^2}\\ \dfrac{\partial^2 v_3}{\partial x_1^2}+\dfrac{\partial^2 v_3}{\partial x_2^2}+\dfrac{\partial^2 v_3}{\partial x_3^2}\end{pmatrix}$$

تمرین: به کمک نمادگذاری شاخصی، حاصل عبارت $grad\,(div\vec{V})$ را به‌دست آورید.